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Stable three-dimensional spatiotemporal solitons in a two-dimensional photonic lattice

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We investigate the existence and stability of three-dimensional spatiotemporal solitons in self-focusing cubic Kerr-type optical media with an imprinted two-dimensional harmonic transverse modulation of the refractive index. We demonstrate that two-dimensional photonic Kerr-type nonlinear lattices can support stable one-parameter families of three-dimensional spatiotemporal solitons provided that their energy is within a certain interval and the strength of the lattice potential, which is proportional to the refractive index modulation depth, is above a certain threshold value.

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Spatiotemporal solitons (STS's) in optical media have attracted much attention in the last several years (see, e.g., [1,2]). A STS, sometimes referred to as a "light bullet" [3], is a nondiffracting and nondispersing wave packet propagating in suitable optically nonlinear material. It retains its shape and is guided along the propagation direction by virtue of a balance among diffraction, group-velocity dispersion, and nonlinear self-phase modulation. Solitons (or, more properly, solitary waves) in Kerr-type focusing media are described by the cubic nonlinear Schrödinger (NLS) equation and they are known to be unstable in two and three dimensions because of the occurrence of the collapse of the wave packet (see, e.g., the review [4]). Various schemes to arrest the collapse were proposed such as the use of weaker saturable [5] or quadratic nonlinearities [6–9], or to use the concept of nonlinearity and group-velocity dispersion (GVD) management in tandem structures, which are composed of periodically alternating linear dispersive and quadratically nonlinear layers [10]. Other recent approaches use off-resonance two-level systems [11], self-induced-transparency media [12], inhomogeneous, dispersive nonlinear media (for example, a graded index Kerr medium [13]), or small negative fourth-order GVD to arrest the spatiotemporal collapse [14] (this scheme works only in two dimensions, that is, in a planar waveguide with pure Kerr nonlinearity). However, a very promising way to arrest the collapse in Kerr-type focusing media is to use two-dimensional (2D) nonlinear photonic lattices [15–19] in a three-dimensional (3D) environment. The study of coherent wave propagation in lattice systems (including soliton phenomena in nonlinear periodic structures) generated in recent years intense activity both in optics (for a recent comprehensive review, see [20]) and in the field of matter waves in optical trapping potentials [21].

It is the aim of this work to study the existence and stability of the families of 3D spatiotemporal solitons trapped in a 2D Kerr-type nonlinear photonic lattice. The basic dimensionless evolution equation for 3D light propagation in a 2D photonic lattice is

$$i\frac{\partial q}{\partial \xi} = -\frac{1}{2}\left(\frac{\partial^2 q}{\partial \eta^2} + \frac{\partial^2 q}{\partial \zeta^2}\right) + \frac{d}{2}\frac{\partial^2 q}{\partial \tau^2} + \sigma q|q|^2 - p\cos(\Omega_\eta \eta)\cos(\Omega_\zeta \zeta)q. \quad (1)$$

Here one assumes that the linear susceptibility depends on the transverse coordinates η and ζ, which finally leads to the appearance of the linear term proportional to the modulation parameter p (see, e.g., [22] and [23] for the normalization of the physical quantities), τ is the time coordinate, and the parameter d equals the ratio of diffraction length to dispersion length. We consider the case of *anomalous* temporal dispersion and we take $d=-1$ because we can always rescale the time coordinate to get this value. The parameter σ defines the sign of the nonlinearity, and we consider the case of self-focusing nonlinearity $(\sigma=-1)$. The parameter p is proportional to the refractive index modulation depth. We consider equal modulation frequencies $\Omega_\eta=\Omega_\zeta=2\pi/T$, where T is the modulation period. Equation (1) conserves the energy $E=\int\int\int|q(\eta,\zeta,\tau)|^2 d\eta d\zeta d\tau$, and the Hamiltonian

$$H=\int\int\int\left[\frac{1}{2}\left(\left|\frac{\partial q}{\partial \eta}\right|^2+\left|\frac{\partial q}{\partial \zeta}\right|^2+\left|\frac{\partial q}{\partial \tau}\right|^2\right) - \frac{1}{2}|q|^4 - p\cos(\Omega_\eta\eta)\cos(\Omega_\zeta\zeta)|q|^2\right]d\eta d\zeta d\tau. \quad (2)$$

We search for stationary soliton profiles in the form $q(\eta,\zeta,\tau,\xi)=w(\eta,\zeta,\tau)\exp(ib\xi)$, where w is a real function and b is the nonlinear wave number shift. The lattice soliton families are defined by the propagation constant b, modulation period T, and the guiding parameter p. Since the scaling transformation $q'(\eta,\zeta,\tau,\xi,p)=\chi q(\chi\eta,\chi\zeta,\chi\tau,\chi^2\xi,\chi^2 p)$ can be used to obtain various families of lattice solitons from a given one, we selected the transverse scale in such a way that the modulation period $T=\pi/2$ and we varied b and p [22]. The resulting equation was solved by using the imaginary time propagation method (see, e.g., [24]). We have used a standard Crank-Nicholson finite difference scheme. The nonlinear finite-difference equations were solved by means

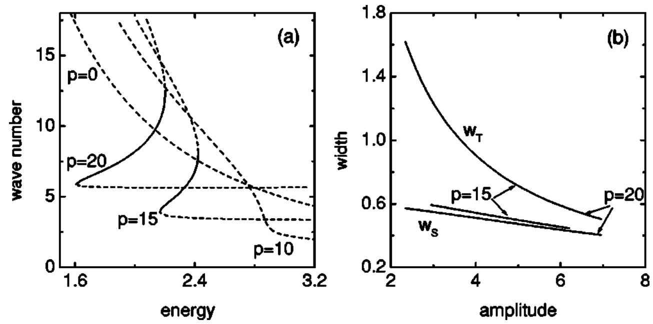

FIG. 1. (a) The wave number b vs energy E. (b) The temporal (w_T) and spatial (w_S) widths vs soliton's amplitude A. Solid lines show stable solitons whereas dashed lines show unstable ones. In (b) only the stable branches were plotted.

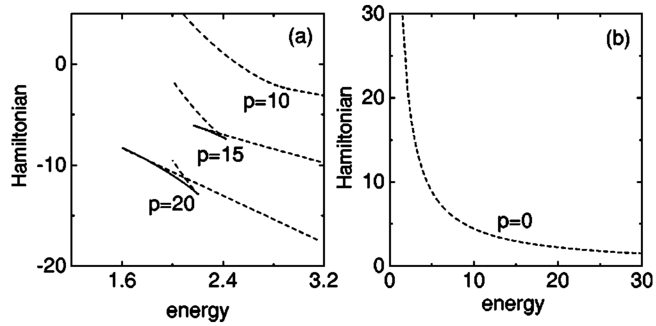

FIG. 2. (a) The Hamiltonian H of the 3D solitons vs their energy E for $p \neq 0$. (b) The $H=H(E)$ plot for the case $p=0$. Solid lines show stable solitons, whereas dashed lines show unstable ones.

of the Picard iteration method and the resulting linear system was handled with the help of the Gauss-Seidel iterative procedure. To achieve good convergence, we needed typically, six Picard iterations and six Gauss-Seidel iterations. We have employed equal transverse grid step sizes $h=\Delta\eta=\Delta\zeta=\Delta\tau$. Typical values of the transverse step sizes and longitudinal step sizes are $h=0.008$ and $\Delta\xi=0.000\,06$ for high amplitude solitons. For low amplitude solitons we take typically $h=0.05$ and $\Delta\xi=0.002$. We have used 401 points in each transverse direction and the stabilization process occurs after $4\times10^3-1\times10^4$ steps along the propagation direction.

By a direct manipulation of the evolution equation (1) we get the following relationship between the total energy E, the wave number b, the stationary profile w, and the Hamiltonian H: $H=-bE+\frac{1}{2}\int\int\int w^4 d\eta d\zeta d\tau$. This relationship may be used to determine the wave number b, once knowing the field profile w. Notice that for the stationary solitons of the 3D NLS equation [that is, the limit $p=0$ in Eq. (1)] the corresponding relationships between E, H, and the wave number b are $b(E)=CE^{-2}$ and $H(E)=CE^{-1}$, where $C\simeq 44.3$ is a numerical constant (see, e.g., [1,2]). We mention that we have additionally cross-checked our imaginary time propagation code on the 3D NLS equation and we have tested the validity of the above relationships between the wave number b, the energy E, and the Hamiltonian H.

In Fig. 1 we plot the dependence $b=b(E)$ [Fig. 1(a)] and the spatial (w_S) and temporal (w_T) widths vs soliton's amplitude A [Fig. 1(b)] for the one-parameter family of 3D stationary solutions. We see that the 3D solitons are isotropic in the two confining directions [in the plane (η,ζ)] and are elongated in the time coordinate τ; for low soliton's amplitudes the corresponding temporal width w_T is more than two times larger than the associated spatial width w_S [see Fig. 1(b)]. As a consequence of the imprinted 2D photonic lattice, the nonlinear localized states exist only for wave numbers b larger than some minimum values $b_{min}(p)$ (the edge of the band gap). This minimum propagation constant increases with the increase of the lattice strength parameter p [for the NLS equation ($p=0$) we have $b_{min}=0$]. Families of 3D solitons in a 2D lattice exist whenever their energy exceeds a certain minimum value and are linearly stable in the intermediate-energy regime and for sufficiently high lattice potential [see the full lines in Fig. 1(a) for lattice strengths $p=15$ and $p=20$]. For the sake of comparison we plotted also in Fig. 1(a) and Fig. 2(b) the dependences $b=b(E)$ and $H=H(E)$, respectively, for the unstable solitons corresponding to the 3D NLS equation ($p=0$). Remarkably, for sufficiently large values of the lattice strength parameter p, the Hamiltonian-versus-energy curves plotted in Fig. 2(a) display *two cusps*, instead of a single one as in other 2D and 3D Hamiltonian systems (see, e.g., Ref. [2] for several examples of the usefulness of the Hamiltonian-versus-energy diagrams in the analysis of the existence and stability of solitons in conservative systems). This feature is intimately connected to the existence of stable 3D solitons within a finite interval of their energies. This two-cusp structure is the so-called "swallow tail" catastrophe and is quite rare in physics (for a review on catastrophe theory as applied to the soliton stability see, e.g., [25]). Notice that a multivalued $H=H(E)$ curve with two cusps was encountered also in the case of one- and two-dimensional type-II quadratic solitons [26]; however, in that case for a given energy there exist two stable solitons and one unstable one; a situation different from that presented in Fig. 2(a). We mention that only the soliton families which meet the Vakhitov-Kolokolov criterion $dE/db>0$ were expected to be linearly stable. We have checked by direct propagation that this is the case: the two semi-infinite branches corresponding to $p=15$ and $p=20$ [see the dashed lines in Figs. 1(a) and 2(a)] correspond to unstable STS's, whereas the finite branches [see the solid lines in Figs. 1(a) and 2(a)] correspond to stable STS's. Thus in sharp contrast to 2D spatial solitons that become stable over certain threshold propagation constant b_{cr} at any lattice depth $p>0$ [23], the 3D solitons supported by 2D lattices can be stable only when the lattice depth p exceeds a certain critical value. Moreover, in contrast to spatial solitons that are stable in a semiinfinite domain $b>b_{cr}$, the STS's can be stable in a limited interval of propagation constants that grows with an increase of lattice depth. Thus, the inclusion of the anomalous temporal dispersion makes it harder to stabilize the 3D solitons (especially narrow ones).

Figures 3(a) and 3(b) illustrate the shape of a typical low amplitude stable 3D lattice soliton whose wave number is close to the band edge. In this case the soliton spreads to several lattice sites; moreover, this low amplitude soliton has an elongated shape along the time coordinate,

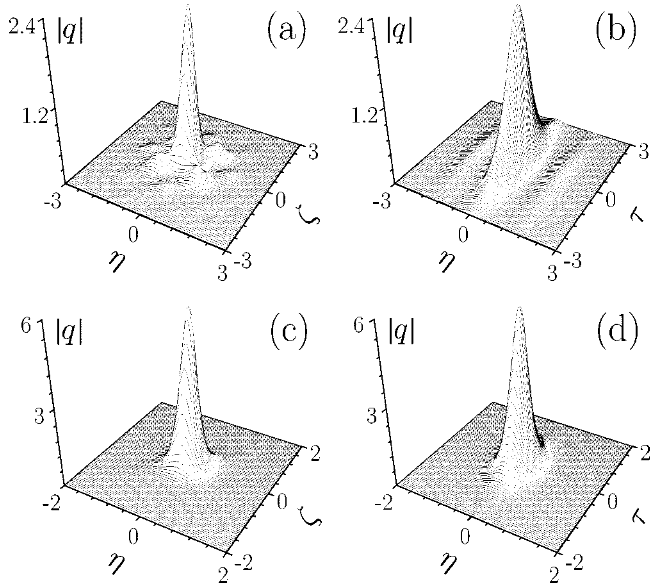

FIG. 3. Cross sections at $\tau=0$ (a) and (c) and $\zeta=0$ (b) and (d) of stable 3D solitons for $p=20$. In (a) and (b) the peak amplitude $A=2.4$, the energy $E=1.6$, and the wave number $b=5.965$, whereas in (c) and (d) $A=6$, $E=2.17$, and $b=10.702$.

in accordance with the amplitude-width dependence plotted in Fig. 1(b). A typical high amplitude single-peaked soliton which occupies a single lattice cell is shown in Figs. 3(c) and 3(d) for the same value of the lattice strength parameter p but for a higher value of the energy and of the propagation constant (near the other side of the stability interval). Thus the temporal and spatial widths of high amplitude 3D lattice solitons are quite close to each other [see Fig. 1(b)].

An important issue for these 3D solitons trapped in a 2D photonic lattice is the occurrence of collapse. We expect that the solitons corresponding to the stable branches [see the solid lines in Figs. 1(a) and 2(a)] are able to withstand small perturbations without collapse, whereas linearly unstable solitons either collapse or spread into linear Bloch waves, depending on the type and strength of the perturbation. In order to confirm these expectations we nonlinearly evolved the stationary solitons under small perturbation, taking the initial condition of the form $q(\xi=0)=w(\eta,\zeta,\tau)(1+\epsilon\rho)$, where ϵ is a small quantity and ρ can be taken either as an uniformly distributed random number in the interval $[-0.5,0.5]$ or as $\rho=1$ (uniform perturbation). We have checked that the solitons on the stable interval of the nonlinear dispersion curve [see the solid lines in Fig. 1(a)] are stable against white-noise perturbations [see Figs. 4(a) and 4(b) and 5(a) and 5(b)] for an example of a linearly stable soliton which resisted an input noise with $\epsilon=0.05$. We have checked that the soliton's amplitude, its temporal and spatial widths oscillate slightly, and no collapse or breakup is observed during propagation. We have also studied the nonlinear evolution process under the second type of perturbation when we simply take $\rho=1$. In this case when the strength of the perturbation is very small (ϵ is of the order of 0.01), the perturbed soliton slightly oscillates around its stable state, meaning that the linearly stable solitons are also

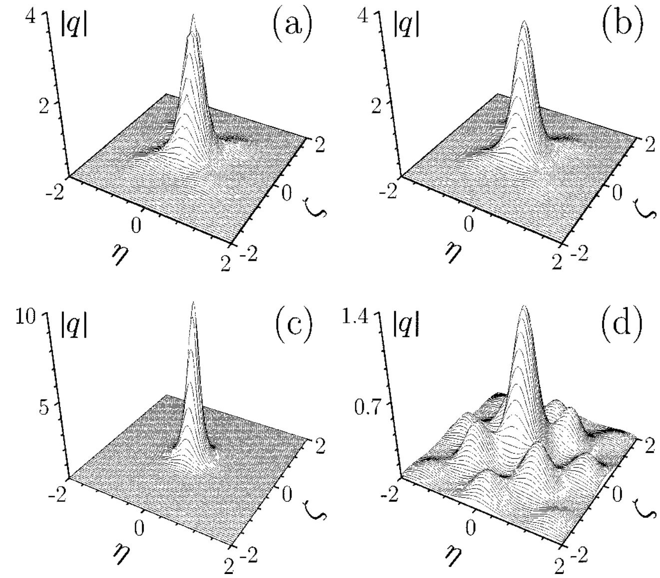

FIG. 4. Cross sections at $\tau=0$ of two solitons corresponding to the same energy $E=2.2$ and for $p=15$. (a) Input white-noise perturbed soliton, (b) the self-cleaned stable soliton at $\xi=20$, (c) input high amplitude unstable soliton, and (d) the unstable soliton at $\xi=2$.

nonlinearly stable. For the linearly unstable solitons we have identified two different instability scenarios: they either decay into linear Bloch waves under white-noise perturbations [see Figs. 4(c) and 4(d) and 5(c) and 5(d) for a typical situation], or they collapse if the soliton is perturbed by a strong uniform perturbation with the strength $\epsilon=0.1$. However, the collapse is not encountered in the case of spatial solitons supported by 2D lattices: the 2D solitons transform into either linear Bloch waves or into narrower solitons belonging to the stable branch [23].

We note that our results concerning the existence and sta-

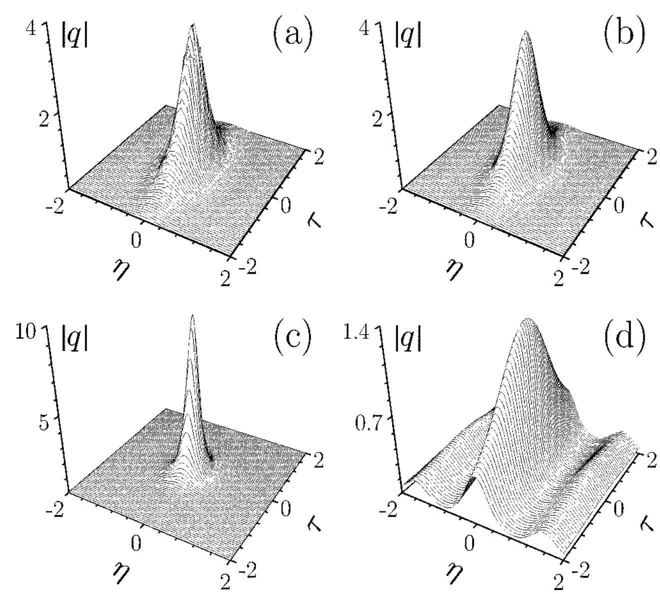

FIG. 5. The same as in Fig. 4, but for the cross sections at $\zeta=0$.

bility of the one-parameter family of multidimensional solitons in a low-dimensional photonic lattice might be useful for other branches of physics, such as the nonlinear dynamics of a Bose-Einstein condensate loaded in an optical lattice [19,27–32].

In conclusion, we have found one parameter families of three-dimensional spatiotemporal solitons in a confining two-dimensional Kerr-type photonic lattice. The stable low-energy (amplitude) solitons whose propagation constants are close to the band edge have a composite multicell structure (that is, the soliton is composed of a central peak with symmetrically spaced satellites in neighboring cells of the optical lattice), whereas the stable high-energy (amplitude) solitons have a single-cell structure, being largely confined to only one lattice site. The three-dimensional solitons undergo abrupt delocalization as the strength of the two-dimensional photonic lattice is decreased below some critical value, whereas for sufficiently high lattice strengths, they are both linearly and nonlinearly stable in a finite interval of their energies.

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